Indian J. Phys. **54B**, 540–546 (1980)

A Bianchi type I viscous fluid cosmological model in general relativity

S R Roy and O P Tiwari

Department of Mathematics, Banaras Hindu University, Varanasi 221005

Received 1 November 1978, revised 18 June 1979

Abstract : A Bianchi type I viscous fluid cosmological model has been derived by using a supplementary condition between the metric potentials.

Roy and Prakash (1976) have derived some viscous fluid cosmological models of plane symmetry in which the free gravitational field is of Petrov type D. A gravitationally non degenerate viscous fluid cosmological model of cylindrical symmetry has also been derived by Roy and Prakash (1977). In this paper we have obtained a viscous fluid cosmological model of Bianchi type I by using a supplementary condition between the metric potentials. The model represents a universe that starts expanding at time $\tau = 0$ from a singular state and goes over to flat space time at time $\tau = \infty$. The model reduces to that of perfect fluid in the absence of viscosity. Reality conditions involving pressure and density have been discussed.

We consider the metric in the form

$$ds^2 = A^2(dx^2-dt^2)+B^2dy^2+C^2dz^2 \tag{1.1}$$

where the metric potentials A, B and C are functions of time alone. The energy momentum tensor for viscous fluid distribution is given by

$$\begin{aligned}T^j_i = (\epsilon+p)V_iV^j+pg^j_i+\eta(V^j{}_{,i}+V^j{}_{;i}+V^jV^lV_{i;l}\\ +V_iV^l;V^j{}_{;l})-(\rho-2/3\,\eta)V^l{}_{,l}(g^j_i+V_iV^j)\end{aligned} \tag{1.2}$$

together with

$$V_iV^i = -1, \tag{1.3}$$

p being the isotropic pressure, ϵ the density, η and ρ the two coefficients of viscosity and ; indicates co-variant differentiation. V^i is the flow vector satisfying (1.3). We assume the co-ordinates to be comoving so that

$$V^1 = V^2 = V^3 = 0 \quad \text{and} \quad V^4 = \frac{1}{A}$$

The field equations

$$-8\pi T^j_i = R^j_i-\tfrac{1}{2}Rg^j_i+\Lambda g^j_i$$

for the line element (1.1) are

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B}-\frac{C_{44}}{C}-\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right]-\Lambda$$

$$=8\pi\left[p+\eta\frac{A_4}{A^2}-\left(\rho-\frac{2}{3}\eta\right)V^l{}_{;l}\right]. \tag{1.4}$$

$$-\frac{1}{A^2}\left[\frac{A_{44}}{A}-\frac{A_4{}^2}{A^2}+\frac{B_{44}}{B}\right]-\Lambda$$

$$=8\pi\left[p-2\eta\frac{C_4}{AC}-\left(\rho-\frac{2}{3}\eta\right)V^l{}_{;l}\right], \tag{1.5}$$

$$-\frac{1}{A^2}\left[\frac{A_{44}}{A}-\frac{A_4{}^2}{A^2}+\frac{C_{44}}{C}\right]-\Lambda$$

$$=8\pi\left[p-2\eta\frac{B_4}{AB}-\left(\rho-\frac{2}{3}\eta\right)V^l{}_{;l}\right], \tag{1.6}$$

$$\frac{1}{A^2}\left[\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right]+\Lambda=8\pi\epsilon \tag{1.7}$$

The suffix 4 after the symbols A, B and C indicates ordinary differentiation with respect to time.

2. Solution of the field equations

Equations (1.4)–(1.7) are four equations in five unknowns A, B, C, ϵ and p For complete solution of these four equations we need an extra condition. We assume that

$$A=B^nC^n. \tag{2.1}$$

From equations (1.5) and (1.6) we have

$$\left(\frac{\nu_4}{\nu}\right)_4+\frac{\mu_4}{\mu}\left(\frac{\nu_4}{\nu}\right)=-16\pi\eta\mu^n\left(\frac{\nu_4}{\nu}\right) \tag{2.2}$$

and from equations (1.4), (1.5) and (2.1) we obtain

$$\left(\frac{\nu_4}{\nu}\right)_4+\frac{\mu_4}{\mu}\left(\frac{\nu_4}{\nu}\right)+(2n-1)\frac{\mu_{44}}{\mu}$$

$$=-16\pi\eta\mu^n\left[(2n-1)\frac{\mu_4}{\mu}+\frac{\nu_4}{\nu}\right] \tag{2.3}$$

where $BC=\mu$ and $B/C=\nu$. From equations (2.2) and (2.3), we

$$\mu_{44}+16\pi\eta\mu^n\mu_4=0. \tag{2.4}$$

Case Ia : From equation (2.4) we get for $n \neq -1$,

$$\frac{d\mu}{dt} = b - \frac{16\pi\eta}{n+1}\mu^{n+1} \tag{2.5}$$

b being constant of integration. From equations (2.5) and (2.2) we obtain

$$\nu = \delta\mu^{-\beta(n+1)} \tag{2.6}$$

where δ and β are constants of integration. By suitable transformation of co-ordinates the line element (1.1) takes the form

$$ds^2 = \left[\frac{K}{m^2}\left(1-e^{-m^2T}\right)\right]^{2(1-\alpha)} dX^2 + \left[\frac{K}{m^2}\left(1-e^{-m^2T}\right)\right]^{\alpha-\beta} dY^2 + \left[\frac{K}{m^2}\left(1-e^{-m^2T}\right)\right]^{\alpha+\beta} dZ^2 - dT^2 \tag{2.7}$$

where $m^2 = 16\pi\eta$, $\alpha = \frac{1}{n+1}$, $b = K\alpha$ and $\mu^{n+1} = \frac{K}{m^2}\left(1-e^{-m^2T}\right)$

Case Ib : When $n = -1$ the metric (1.1) takes the form

$$ds^2 = T^{2}\left[dX^2 - \frac{dT^2}{\{\log(\gamma T^{-m^2})\}^2}\right] + T^{1+M}dY^2 + T^{1-M}dZ^2 \tag{2.8}$$

after suitable transformation of co-ordinates, γ and M being constants of integration and $\mu = T$.

Case II Let $\mu = N$ (a constant). The metric (1.1) in this case takes the form

$$ds^2 = N^{2n}(dx^2 - dt^2) + N\nu dy^2 + N\nu^{-1}dZ^2 \tag{2.9}$$

where

$$\log \nu = \frac{e^k\left(l^{-m^2N^nt} - 1\right)}{-m^2N^n} + d$$

k and d being arbitrary constants.

3. Some physical and geometrical features

The pressure and density for the model (2.7) are given by

$$8\pi p = -\frac{1}{4}(3\alpha^2 - 4\alpha + \beta^2)m^4\left(l^{m^2T} - 1\right)^{-2} + \left(8\pi\rho\alpha + \frac{2}{3}m^2\right)m^2\left(e^{m^2T} - 1\right)^{-1} - \Lambda, \tag{3.1}$$

$$8\pi\epsilon = -\frac{1}{4}(3\alpha^2 - 4\alpha + \beta^2)m^4\left(e^{m^2T} - 1\right)^{-2} + \Lambda. \tag{3.2}$$

The scalar of expansion $\theta = V^i{}_{;i}$ for the flow vector V^i is given by

$$\theta = m^2(e^{m^2T}-1)^{-1}. \tag{3.3}$$

The expansion is positive when $\tau > 0$. The rotation ω_{ij} is identically zero and the shear is given by

$$\sigma^2 = \frac{1}{12}[(3\alpha-2)^2+3\beta^2]m^4(e^{m^2T}-1)^{-2}, \tag{3.4}$$

The red shift in the model (2.7) is given by

$$\frac{\lambda+d\lambda}{\lambda} = \frac{\left[\left\{\frac{K}{m^2}(1-e^{-m^2T_1})\right\}^{-\frac{\alpha+\beta}{2}}+U_z\right]}{\left\{\frac{K}{m^2}(1-e^{-m^2T_2})\right\}^{-\frac{\alpha+\beta}{2}}[1-U^2]^{\frac{1}{2}}} \tag{3.5}$$

where U is the velocity of source at the time of emission and O_z is the z component of velocity. The non-vanishing components of conformal curvature tensor are given by

$$C_{12}{}^{12} = \frac{m^4}{12}[(3\alpha+3\beta-2)(e^{m^2T}-1)^{-1}$$

$$+(3\alpha^2-\beta^2-6\alpha\beta-2\alpha+6\beta)(e^{m^2T}-1)^{-2}] \tag{3.6}$$

$$C_{13}{}^{13} = \frac{m^4}{12}[(3\alpha-3\beta-2)(e^{m^2T}-1)^{-1}$$

$$+(3\alpha^2-\beta^2+6\alpha\beta-2\alpha-6\beta)(e^{m^2T}-1)^{-2}], \tag{3.7}$$

$$C_{23}{}^{23} = \frac{m^4}{6}[(2-3\alpha)(e^{m^2T}-1)^{-1}+(\beta^2-3\alpha^2+2\alpha)(e^{m^2T}-1)^{-2}]. \tag{3.8}$$

Hence the space time is non-degenerate Petrov type I unless $\beta = 0$, in which case it is type D. In the absence of viscosity

$$\sigma^2 = \frac{1}{12}\left[\frac{(3\alpha-2)^2+3\beta^2}{T^2}\right]. \tag{3.9}$$

Thus we find that in the absence of viscosity the shear decreases to zero as $1/T^2$. The viscous term causes exponential decrease in the shear in agreement with the result of Misner (1968). Similarly the expansion θ and conformal curvature tensor decrease exponentially with time T. Thus although these quantitie decrease with time T in the absence of viscosity, the effect of viscosity is to hasten this process. If l be the linear dimension of the universe we find from the relation

$$\frac{1}{l} l_{,a} V^a = \frac{1}{3}\theta \tag{3.10}$$

that

$$l = \xi m^{-2/3}(1-e^{-m^2T})^{1/3} \tag{4.11}$$

where ξ is positive constant. When $T = 0$, $l = 0$. We also find that the deceleration parameter q defined by

$$q = -\frac{(l_{,a}V^a)_{,\beta}V^\beta l}{(l_{,a}V^a)^2} \tag{3.12}$$

is given by

$$q = (3e^{m^2T}-1) \tag{3.13}$$

since $q > 0$ the universe had a singularity in the past, viz. at $T = 0$ (Ellis 1971). Hence the model starts expanding from its singular state at time $T = 0$ and continuous to expand till $T = \infty$, at which stage the space time becomes flat, the metric assuming the form

$$ds^2 = \left(\frac{K}{m^2}\right)^{2(1-\alpha)} dX^2 + \left(\frac{K}{m^2}\right)^{\alpha-\beta} dY^2 + \left(\frac{K}{m^2}\right)^{\alpha+\beta} dZ^2 - dT^2. \tag{3.14}$$

The temporal history of the model does not, however, span the entire time period $0 < T < \infty$ since it is restricted by the reality conditions involving the density and pressure. For a realistic distribution we require that the density be positive, the pressure be non negative and that the velocity of sound does not exceed the velocity of light in the medium. These conditions are respectively

(i) $\epsilon > 0$

(ii) $p \geqslant 0$

and

(iii) $\epsilon \geqslant p$.

For the model (2.7) these imply

$$-\frac{1}{4}(3\alpha^2-4\alpha+\beta^2)m^4(e^{m^2T}-1)^{-2}+\Lambda \tag{3.15}$$

$$-\frac{1}{4}(3\alpha^2-4\alpha+\beta^2)m^4(e^{m^2T}-1)^{-2}$$

$$+(8\pi\rho\alpha+\frac{2}{3}m^2)m^2(e^{m^2T}-1)^{-1}-\Lambda \geqslant 0, \tag{3.16}$$

$$-(8\pi\rho\alpha+\frac{2}{3}m^2)m^2(e^{m^2T}-1)^{-1}+2\Lambda \geqslant 0. \tag{3.17}$$

Taking $\alpha > 0$ we find that the conditions are not satisfied for $\Lambda \leqslant 0$. However if $\Lambda > 0$ we find from (3.15), (3.16) and (3.17) that

$$1+\frac{m^2}{2\Lambda}(8\pi\rho\alpha+\frac{2}{3}m^2) \leqslant e^{m^2T} \leqslant 1+\frac{m^2}{2\Lambda}\Big\{(8\pi\rho\alpha+\frac{2}{3}m^2)$$

$$+\sqrt{(8\pi\rho\alpha+\frac{2}{3}m^2)^2-\Lambda(3\alpha^2-4\alpha+\beta^2)}\Big\} \tag{3.18}$$

for both positive as well as negative values of $3\alpha^2-4\alpha+\beta^2$. In the former case we also require that

$$(8\pi\rho\alpha+\frac{2}{3}m^2)^2 > \Lambda(3\alpha^2-4\alpha+\beta^2). \tag{3.19}$$

We may replace (iii) by the more stringent condition

(iv) $\epsilon \geqslant 3p$

which for the model (2.7) takes the form

$$\tfrac{1}{3}(3\alpha^2-4\alpha+\beta^2)m^4(e^{m^2T}-1)^{-2}$$

$$-(24\pi\rho\alpha+2m^2)m^2(e^{m^2T}-1)^{-1}+4\Lambda > 0 \tag{3.20}$$

From (3.15), (3.16) and (3.20) we obtain the limits within which T must lie, viz that

$$1+\frac{m^2}{2}\sqrt{\frac{3\alpha^2-4\alpha+\beta^2}{\Lambda}} < e^{m^2T} < 1+\frac{3}{4}\cdot\frac{m^2}{2\Lambda}\Big\{(8\pi\rho\alpha+\frac{2}{3}m^2)$$

$$+\sqrt{(8\pi\rho\alpha+\frac{2}{3}m^2)^2-\frac{8}{9}\Lambda(3\alpha^2-4\alpha+\beta^2)}\Big\} \tag{3.21}$$

when $\Lambda > 0$ and $3\alpha^2-4\alpha+\beta^2 > 0$; and

$$1+\frac{3}{4}\frac{m^2}{2\Lambda}\Big\{8\pi\rho\alpha+\frac{2}{3}m^2$$

$$-\sqrt{(8\pi\rho\alpha+\frac{2}{3}m^2)^2-\frac{8}{9}\Lambda(3\alpha^2-4\alpha+\beta^2)}\Big\}$$

$$\leqslant e^{m^2T} < 1+\frac{3}{4}\frac{m^2}{2\Lambda}\Big\{(8\pi\rho\alpha+\frac{2}{3}m^2$$

$$+\sqrt{(8\pi\rho\alpha+\frac{2}{3}m^2)^2-\frac{8}{9}\Lambda(3\alpha^2-4\alpha+\beta^2)}\Big\}, \tag{3.22}$$

when $\Lambda > 0$ and $3\alpha^2-4\alpha+\beta^2 < 0$.

In the absence of viscosity the metric (2.7) takes the form

$$ds^2 = (KT)^{2(1-\alpha)}dX^2+(KT)^{\alpha-\beta}dY^2+(KT)^{\alpha+\beta}dZ^2-dT^2$$

for which (3.23)

$$8\pi\alpha = -\frac{1}{4}(3\alpha^2-4\alpha+\beta^2)^{-2}T-\Lambda, \tag{3.24}$$

$$; = -\frac{1}{4}(3\alpha^2-4\alpha+\beta^2)^{-2}T+\Lambda \tag{3.25}$$

The reality conditions (i), (ii) and (iii) are satisfied when $\Lambda \geqslant 0$ and $3\alpha^2-4\alpha+\beta^2 < 0$ and in the case $\Lambda > 0$ we have

$$T^2 < -\frac{(3\alpha^2-4\alpha+\beta^2)}{4\Lambda} \tag{3.26}$$

For the metrics (2.8) and (2.9). the reality conditions are not satisfied,

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